Variational Formulation and Discretization of Multi-Body-Systems with Fluid-Structure Interaction at Low Reynolds Number

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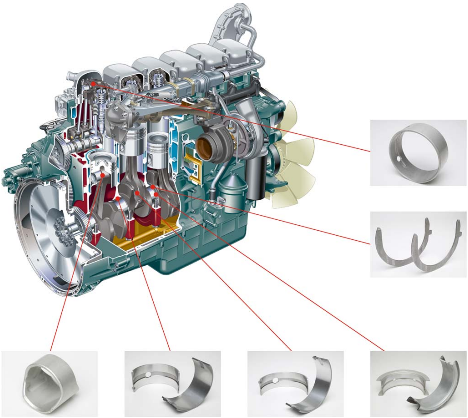
Motivation

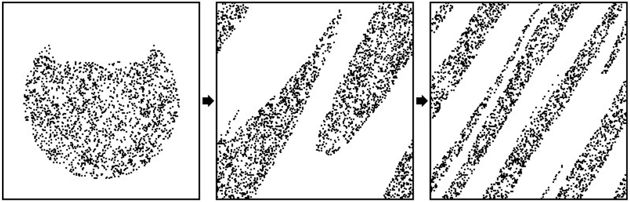
http://www.autolexikon.de



Lubricated contacts are common in machines and mechanisms.

Motivation

$$
\begin{aligned}
0 &= D_1 L_d(q_k, q_{k+1}) + f_d^-(q_k, q_{k+1}) + p_k \\
p_{k+1} &= D_2 L_d(q_k, q_{k+1}) + f_d^+(q_k, q_{k+1})
\end{aligned}
$$



Variational formulations are a good point of departure for numerical methods, such as variational integrators.

Outline and Literature behind

- Modelling Overview [H. Stone, J. Donea & A. Huerta, J. Wauer, B. Schweizer]
- Variational Formulation [B. Finlayson, J. Donea & A. Huerta]
- Variational Discretization [E. Trefftz, L. Collatz]
- Minimal Example

State of the Art

- Simplifications lead via the Reynolds-Equation to a generalized force element
  - low-dimensional discretization
  - fixed precision
- *WANTED*
  - low-dimensional discretization (preferably variational)
  - tunable precision (within modeling assumptions)
- CFD Finite-Elements (e.g. Taylor-Hood)
  - high-dimensional discretization (gap geometry)
  - tunable precision (mesh size)

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Modelling Assumptions

The fluid is modeled, assuming

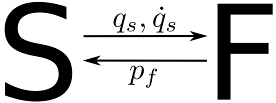
- an incompressible Newtonian Fluid with
- dominating viscous forces ($\mathrm{Re} \ll 1$),
- vanishing inertial and body forces,
- no cavitation (may be too hard restriction for journal bearings).

The fluid flow is described by Stokes Equations

$$
\begin{aligned}
0 &= -\nabla p + \mu \nabla^2 \mathbf{v} && \text{in } \Omega && \text{(equilibrium)},\\
0 &= \nabla \cdot \mathbf{v} && \text{in } \Omega && \text{(incompressibility)},\\
\mathbf{v}_D &= \mathbf{v} && \text{on } \Gamma_D && \text{(Dirichlet B.C.)},\\
\mathbf{t} &= -p\mathbf{n} + \mu(\mathbf{n} \cdot \nabla)\mathbf{v} && \text{on } \Gamma_N && \text{(Neumann B.C.)}.
\end{aligned}
$$

Fluid-Structure-Interaction

- Fluid velocity is determined on boundary by the no-slip condition.
- Fluid pressure acts on structure.



Interaction is typically evaluated during a time step, e.g. at mid-point $t = t_{k+\frac{1}{2}}$.

Variational Formulation

Multiplying the strong form (Laplace formulation for the viscous term) by test functions

$$0 = \int_\Omega (-\nabla p + \mu \nabla^2 \mathbf{v}) \cdot \mathbf{w} + (\nabla \cdot \mathbf{v}) q \, \mathrm{d}\Omega - \int_{\Gamma_N} \left(-p\mathbf{n} + \mu(\mathbf{n} \cdot \nabla)\mathbf{v} - \mathbf{t}\right) \cdot \mathbf{w} \, \mathrm{d}\Gamma$$

and integrating by parts, the stress term

$$0 = \int_\Omega \mu \nabla \mathbf{v} : \nabla \mathbf{w} - p \nabla \cdot \mathbf{w} - q \nabla \cdot \mathbf{v} \, \mathrm{d}\Omega - \int_{\Gamma_N} \mathbf{w} \cdot \mathbf{t} \, \mathrm{d}\Gamma$$

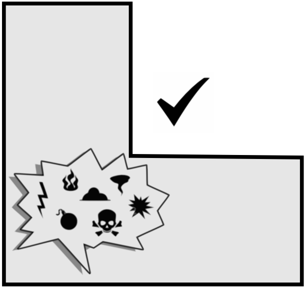
results in the weak form, corresponding to

$$0 = \delta \int_\Omega \frac{1}{2} \mu \nabla \mathbf{v} : \nabla \mathbf{v} - p \nabla \cdot \mathbf{v} \, \mathrm{d}\Omega - \delta \int_{\Gamma_N} \mathbf{v} \cdot \mathbf{t} \, \mathrm{d}\Gamma.$$

Discretization Strategies

Ritz Method[1] (Galerkin similarly)

- boundary values satisfied
- approximation over domain



[1] W. Ritz, 1909: Über eine neue Methode zur Lösung gewisser Variationsprobleme…

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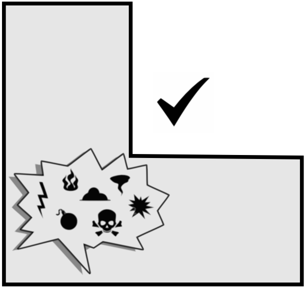
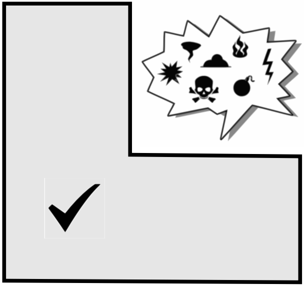


Trefftz Method[2]

- approximation over boundary
- PDE over domain satisfied



[1] W. Ritz, 1909: Über eine neue Methode zur Lösung gewisser Variationsprobleme…

[2] E. Trefftz, 1926: Ein Gegenstück zum Ritzschen Verfahren.

Trefftz Method I/II

A general linear PDE

$$L[u] = r(x,y) \qquad \text{with} \qquad u = g(s) \ \text{on} \ \Gamma$$

is satisfied by the combination $\quad \bar{u} = \bar{u}_0 + \sum_{n=1}^{N} c_n \bar{u}_n$

of a particular solution

$$L[\bar{u}_0] \quad = \quad r$$

and linearly independent solutions of the homogeneous equation

$$L[\bar{u}_n] \quad = \quad 0 \qquad \text{for } \ n = 1 \ldots N.$$

The coefficients c_n are determined by a best fit on the boundary.

Trefftz Method II/II

The error between true solution u and approximation $\bar{u}$ is minimized in terms of the variational formulation

$$J[\bar{u}-u] = \min$$

with necessary minimum condition

$$\frac{\partial}{\partial c_n} J[\bar{u}-u] = 2J[\bar{u}-u, \bar{u}_n] = 0 \qquad \text{for } n = 1 \ldots N.$$

This domain integral may be transformed into a boundary integral by Green's Formula

$$J[\bar{u}-u, \bar{u}_n] = \int_\Omega (\bar{u}-u) L[\bar{u}_n] \,\mathrm{d}\Omega - \int_\Gamma (\bar{u}-u) L^*[\bar{u}_n] \,\mathrm{d}\Gamma \qquad \text{for } n = 1 \ldots N$$

where the domain integral vanishes ($L[\bar{u}_n] = 0$) and the boundary integral

$$\int_\Gamma \left(\bar{u}_0 + \sum_{m=1}^{N} c_m \bar{u}_m - u \right) L^*[\bar{u}_n] \,\mathrm{d}\Gamma \qquad \text{for } n = 1 \ldots N.$$

leads to a linear equation system for the coefficients c_m.

Stokes Flow: Irreducible Formulation I/II

Pressure p (*slave*) depends on velocity v (*master*)

$$\nabla p = \mu \nabla^2 \mathbf{v}.$$

Taking divergence and curl of the equation above gives

$$\begin{aligned} \nabla^2 p &= 0 \quad \text{using incompressibility}, \\ \mu \nabla^2 \boldsymbol{\omega} &= 0 \quad \text{with } \boldsymbol{\omega} = \nabla \times \mathbf{v}. \end{aligned}$$

Enforcing the incompressibility by a stream function

$$v_x = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v_y = -\frac{\partial \psi}{\partial x}$$

results in the following non-zero component of the vorticity vector

$$\mu \nabla^2 \omega_z = \mu \nabla^2 (-\nabla^2 \psi) = 0.$$

Stokes Flow: Irreducible Formulation II/II

The approximative solution is found by solving the PDE

$$\nabla^4 \psi = 0$$

with Trefftz's Method for ψ, from which $\mathbf{v}$ and p follow. Candidates for the composition of an approximation are bi-potential functions

$$\bar{\psi} = x, x^2, x^3, y, y^2, y^3, xy, x^2y, x^3y, xy^2, xy^3, \sin kx \sinh ky, x \sin kx \sinh ky, \ldots$$

The variational form of the bi-potential equation reads

$$J[\psi] = \int_\Omega \frac{1}{2} (\nabla^2 \psi)^2 \, \mathrm{d}A + \int_\Gamma (\psi - \gamma_1) \nabla(\nabla^2 \psi) \cdot \mathbf{n} \, \mathrm{d}s - \int_\Gamma (\nabla \psi \cdot \mathbf{n} - \gamma_2) \nabla^2 \psi \mathbf{n} \, \mathrm{d}s$$

and results in Trefftz's Equations ($m = 1, 2, \ldots, N$)

$$\sum_{n=1}^{N} c_n \int_\Gamma \bar{\psi}_n \nabla(\nabla^2 \bar{\psi}_m) \cdot \mathbf{n} - \nabla \bar{\psi}_n \cdot \mathbf{n} \nabla^2 \bar{\psi}_m \, \mathrm{d}s = \int_\Gamma \gamma_1 \nabla(\nabla^2 \bar{\psi}_m) \cdot \mathbf{n} - \gamma_2 \nabla^2 \bar{\psi}_m \, \mathrm{d}s.$$

Stokes Flow: Mixed Formulation I/III

Consider pressure p and velocity $\mathbf{v}$ as independent fields

$$\begin{aligned} 0 &= -\nabla p + \mu \nabla^2 \mathbf{v}, \\ 0 &= \nabla \cdot \mathbf{v}. \end{aligned}$$

Taking the divergence of the momentum equation using the continuity equation gives (as previously)

$$\nabla^2 p = 0,$$

i.e. the pressure is harmonic and candidates are

$$\bar{p} = x, y, xy, x^2 - y^2, x^3 - 3xy^2, \sin kx \sinh ky, \sinh kx \sin ky, \ldots$$

Note the identity (to be used next)

$$\nabla^2 (p\mathbf{r}) = 2\nabla p + \mathbf{r} \underbrace{\nabla^2 p}_{=0} = 2\nabla p.$$

Stokes Flow: Mixed Formulation II/III

A convenient decomposition is

$$\mathbf{v}(\mathbf{r}) = \frac{1}{2\mu} p\mathbf{r} + \mathbf{v}^h(\mathbf{r}),$$

since the momentum equation (remember $\nabla^2(p\mathbf{r}) = 2\nabla p$)

$$-\nabla p + \mu\nabla^2 \left(\frac{1}{2\mu} p\mathbf{r} + \mathbf{v}^h(\mathbf{r}) \right) = \mu\nabla^2\mathbf{v}^h = 0$$

reduces to Laplace Equation for $\mathbf{v}^h \approx \bar{\mathbf{v}}^h = \bar{\mathbf{v}}_0^h + \sum_{n=1}^{N_v} \bar{\mathbf{v}}_n^h$.

Potential functions are appropriate candidates with regard to Trefftz Method, however the boundary conditions for $\mathbf{v}^h$ can not yet be specified without pressure field p.

Stokes Flow: Mixed Formulation III/III

The additional equations to determine $\bar{p} = \bar{p}_0 + \sum_{n=1}^{N_p} \bar{p}_n$ follow from the continuity equation

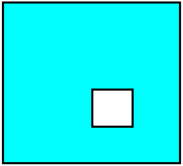
$$0 = \nabla \cdot \left(\frac{1}{2\mu} \bar{p} \mathbf{r} + \bar{\mathbf{v}}^h(\mathbf{r}) \right) .$$

We enforce the incompressibility constraint in a weighted integral sense (Galerkin)

$$0 \quad = \quad \int_\Omega \nabla \cdot \left(\frac{1}{2\mu} \bar{p} \mathbf{r} + \bar{\mathbf{v}}^h \right) w_m \, \mathrm{d}A \qquad \text{for } \; m = 1, 2, \ldots, N_p .$$

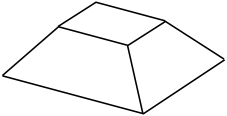
Finally the N_v Trefftz Equations and the N_p Galerkin Equations constitute the system of equations for the $N_v + N_p$ unknown coefficients in the approximations $\bar{\mathbf{v}}^h$ and $\bar{p}$.

Rectilinear Minimal Model I/II



A rectangular body (without rotation) in a rectangular cavity filled with a viscous liquid.

The boundary values correspond to a rigid body motion with velocity $\mathbf{v} = V_y \mathbf{e}_y$.



Note that it is involved to satisfy the Dirichlet Boundary Conditions (rigid body motion) by differentiable functions that were needed for Ritz's Method.

Rectilinear Minimal Model II/II

For the irreducible method the stream function (remember $v_x = \psi_{,y},\ v_y = -\psi_{,x}$) and its directional derivatives (external normal $\mathbf{n}$) need to be specified.
On the fixed boundaries

$$\begin{aligned} \psi &= C_0 = \text{const.} \qquad (v_x = v_y = 0), \\ \nabla\psi \cdot \mathbf{n} &= 0. \end{aligned}$$

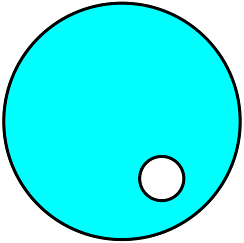
On the moving horizontal boundaries ($\mathbf{n} = \pm\mathbf{e}_y$)

$$\begin{aligned} \psi &= C_1 - V_y x \qquad (v_x = 0,\ v_y = V_y), \\ \nabla\psi \cdot \mathbf{n} &= 0 \qquad (\psi_{,y} = v_x = 0)\,. \end{aligned}$$

On the moving vertical boundaries ($\mathbf{n} = \pm\mathbf{e}_x$)

$$\begin{aligned} \psi &= C_1 - V_y x \qquad (v_x = 0,\ v_y = V_y), \\ \nabla\psi \cdot \mathbf{n} &= \mp V_y \qquad (\psi_{,x} = -v_y = -V_y)\,. \end{aligned}$$

Curvilinear Minimal Model I/II

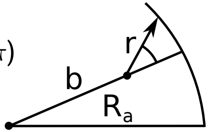


A rotating (prescribed $\alpha(t)$) circular body moving (x, y) in a circular cavity filled with a viscous liquid.

Again (for the irreducible formulation) the stream function and its directional derivative (external normal $\mathbf{n}$) need to be specified on the boundaries (rigid body motion).

Curvilinear Minimal Model II/II

Fixed boundary ($r = -b\cos\varphi + \sqrt{R_a^2 - b^2\sin^2\varphi}$, $-\pi \leq \varphi < \pi$)



$$\begin{aligned} \psi &= C_0 = \text{const.}, \\ \nabla\psi \cdot \mathbf{n} &= 0. \end{aligned}$$

Moving boundary ($r = R_i$, $-\pi \leq \varphi < \pi$)

$$\begin{aligned} \begin{bmatrix} V_x - \dot{\alpha} y \\ V_y + \dot{\alpha} x \end{bmatrix} &= \begin{bmatrix} \psi_{,y} \\ -\psi_{,x} \end{bmatrix} \rightsquigarrow \psi = V_x y - V_y x - \frac{1}{2}\dot{\alpha}(x^2 + y^2), \\ \nabla\psi \cdot \mathbf{n} &= -n_x v_y + n_y v_x = -n_x(V_x - \dot{\alpha} y) + n_y(V_y + \dot{\alpha} x). \end{aligned}$$

Summary

- Stokes Flow shares same mathematical structure with linear elasticity.
- Global approximations are preferred to obtain a low-dimensional discretization of typical geometries.
- Exterior, i.e. boundary, methods, here following Trefftz, are preferred.
- We propose an irreducible and a mixed formulation.

Outlook

- Implementation of the minimal models for verification with textbook models (Stokes Drag, Reynolds-Equation).
- Check for structure-preservation properties of this combination of two variational principles.
- Take advantage of complex analysis for the representation of planar incompressible flow.

Appendix