Variational Integrators and Optimal Control for a Hybrid Pendulum-on-Cart-System

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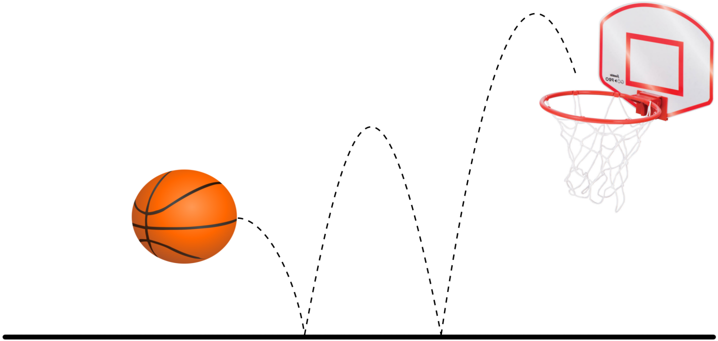
CHEMNITZ

Motivation

How to score, if you have less than the weight as actuation force ($F_{\text{max}} < mg$) ?

Motivation



..by bouncing the ball on the ground, until it reaches the basket.

Motivation

Collisions, e.g. by brakes, may generate forces or torques, much higher than drives of comparable weight and size.

Cubli *– A cube that can jump up, balance, and walk across your desk*
[Institute for Dynamic Systems and Control, ETH Zurich, Switzerland]

Outline of the Talk and Literature behind

- Non-smooth Dynamics
  - C. Glocker
  - B. Brogliato
- Variational Integrators and Optimal Control
  - J.E. Marsden
  - T. Murphey, S. Leyendecker
  - O. Junge, S. Ober-Blöbaum
- Variational Integrators and Optimal Control for Non-smooth Systems
  - J.E. Marsden, R.C. Fetecau
  - T. Murphey, D. Pekarek
  - S. Ober-Blöbaum, K. Flaßkamp
  - S. Leyendecker, M. Koch
- Pendulum on Cart
  - A. Astrom, K. Furuta
  - . . .

Non-smooth Dynamics

Hybrid systems combine time continuous and time discrete dynamics.
In mechanical systems discrete events originate from collision or friction modeling.

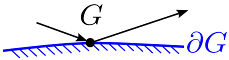
Variational principle in generalized coordinates $q \in \mathbb{R}^n$

$$0 = \delta \int_{t_B}^{t_E} \mathcal{L}(q, \dot{q}, t)\,\mathrm{d}t + \int_{t_B}^{t_E} \underbrace{Q(q, \dot{q}, t) \cdot \delta q}_{\delta W^{\mathsf{nc}}}\,\mathrm{d}t$$

subjected to unilateral constraint

$$g(q) \leq 0$$

admissible positions $q \in G$, collisions at boundary ∂G.



Variational Formulation

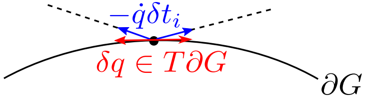
Hamilton's principle for a time interval with a collision at time $t = t_i$, $\quad t_B < t_i < t_E$

$$0 = \delta \int_{t_B}^{t_E} \mathcal{L}\,\mathrm{d}t + \int_{t_B}^{t_E} Q \cdot \delta q\,\mathrm{d}t$$

$$\begin{aligned} 0 &= \int_{t_B}^{t_i^-} \left(\mathcal{L}_q - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\dot q} + Q \right) \cdot \delta q\,\mathrm{d}t + \left| \mathcal{L}_{\dot q} \cdot \delta q + \mathcal{L}\delta t_i \right|_{t_B}^{t_i^-} \\ &+ \int_{t_i^+}^{t_E} \left(\mathcal{L}_q - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{\dot q} + Q \right) \cdot \delta q\,\mathrm{d}t + \left| \mathcal{L}_{\dot q} \cdot \delta q + \mathcal{L}\delta t_i \right|_{t_i^+}^{t_E} + \sigma(t_i)\hat{Q} \cdot \delta q. \end{aligned}$$

Generalized coordinates $q(t)$ are assumed continuous, velocities/momenta not.

Variational Formulation



impact position is restricted to $g\big(q(t_i)\big) = 0$, impact time t_i depends on $q(t)$

$$\begin{aligned}
0 \quad &= \quad \delta g\big(q(t_i)\big) = \nabla g \cdot \big(\delta q(t_i) + \dot{q}(t_i)\delta t_i\big) \\
\delta t_i = 0 \quad &\rightsquigarrow \quad \nabla g \cdot \delta q(t_i) = 0 \rightsquigarrow \delta q(t_i) \in T\partial G \\
\delta t_i \neq 0 \quad &\rightsquigarrow \quad \delta q(t_i) = -\dot{q}(t_i)\delta t_i
\end{aligned}$$

thus boundary term turns for frictionless elastic collisions (smooth Q) into

$$0 = -\left|\mathcal{L}_{\dot{q}} \cdot \delta q + \mathcal{L}\delta t_i\right|_{t_i^-}^{t_i^+} = \begin{cases} -\left|\mathcal{L}_{\dot{q}} \cdot \delta q_{\mathbf{t}}\right|_{t_i^-}^{t_i^+} & \text{for } \delta t_i = 0 \\ -\left|(-\dot{q} \cdot \mathcal{L}_{\dot{q}} + \mathcal{L})\delta t\right|_{t_i^-}^{t_i^+} & \text{otherwise} \end{cases}$$

which represents *conservation of momentum* in tangential directions and *conservation of energy*, respectively.

Variational Integrators (VI)

1.) Approximation of the generalized coordinates per time step $h = t_{k+1} - t_k$

$$q(t) \approx q^d(t) = q_k + \frac{t - t_k}{h}(q_{k+1} - q_k)$$

2.) and numerical integration of the action functional (same for virtual work)

$$\int_{t_k}^{t_{k+1}} \mathcal{L}\big(q(t), \dot{q}(t), t\big)\,\mathrm{d}t \approx h\mathcal{L}\big(q^d(t_{k+1/2}), \dot{q}^d(t_{k+1/2}), t_{k+1/2}\big) = L_d(q_k, q_{k+1}, h)$$

leads to Discrete Euler-Lagrange equations (collision-free case)

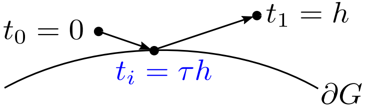
$$\begin{aligned} p_k &= -D_1 L_d(q_k, q_{k+1}, h) - f_d^-(q_k, q_{k+1}, h) \\ p_{k+1} &= D_2 L_d(q_k, q_{k+1}, h) + f_d^+(q_k, q_{k+1}, h). \end{aligned}$$

Notation: $D_1 L_d(q_k, q_{k+1}, h) = \frac{\partial}{\partial q_k} L_d(q_k, q_{k+1}, h)$,

$D_2 L_d(q_k, q_{k+1}, h) = \frac{\partial}{\partial q_{k+1}} L_d(q_k, q_{k+1}, h)$,

$D_3 L_d(q_k, q_{k+1}, h) = \frac{\partial}{\partial h} L_d(q_k, q_{k+1}, h)$.

Collision Time Step



before collision $t_0 \leq t < \tau h \quad \rightsquigarrow \quad q_\tau, \tau$

$$\begin{aligned} \delta q_0 : \; p_0 &= -D_1 L_d(q_0, q_\tau, \tau h) - f_d^-(q_0, q_\tau, \tau h) \\ 0 &= g(q_\tau) \end{aligned}$$

collision at time $t = \tau h \quad \rightsquigarrow \quad q_1,$ remember $\delta q = \delta q_{\mathbf{t}} - \dot{q}\delta t_i$

$$\begin{aligned} \delta q_\tau : \; 0 = \; & \Big(D_2 L_d(q_0, q_\tau, \tau h) + f_d^+(q_0, q_\tau, \tau h) \\ & + D_1 L_d\big(q_\tau, q_1, (1-\tau)h\big) + f_d^-\big(q_\tau, q_1, (1-\tau)h\big) \Big) \cdot \mathbf{t}_i \\ \delta t_\tau : \; 0 = \; & D_3 L_d(q_0, q_\tau, \tau h) - f_d^+(q_0, q_\tau, \tau h) \tfrac{q_\tau - q_0}{\tau h} \\ & - D_3 L_d\big(q_\tau, q_1, (1-\tau)h\big) - f_d^-\big(q_\tau, q_1, (1-\tau)h\big) \tfrac{q_1 - q_\tau}{(1-\tau)h} \end{aligned}$$

after collision $\tau h < t < t_1 \quad \rightsquigarrow \quad p_1$

$$\delta q_1 : \; p_1 = D_2 L_d\big(q_\tau, q_1, (1-\tau)h\big) + f_d^+\big(q_\tau, q_1, (1-\tau)h\big)$$

Discrete Mechanics and Optimal Control (DMOC)

direct approach, local optimization, works for smooth trajectories

$$\begin{aligned}
\min_{u(t),q(t)} J_C &= \int_{t_b}^{t_e} C\big(q(t), \dot{q}(t), u(t)\big)\,\mathrm{d}t && \text{cost functional} \\
\text{s.t.:} \quad q(t_b) &= q_b, \qquad \dot{q}(t_b) = \dot{q}_b && \text{initial conditions} \\
\begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} &= f\big(q(t), \dot{q}(t), u(t)\big) && \text{system dynamics} \\
r\big(q(t), u(t)\big) &\geq 0 \qquad \forall t \in [t_b, t_e] && \text{path constraints} \\
q(t_e) &= q_e, \qquad \dot{q}(t_e) = \dot{q}_e && \text{terminal conditions}
\end{aligned}$$

Discrete Mechanics and Optimal Control (DMOC)

DMOC uses the same discretization for the cost functional as for the system dynamics

$$\begin{aligned} C_d(q_k, q_{k+1}, u_k, u_{k+1}) &\approx \int_{t_k}^{t_{k+1}} C\big(q^d(t), \dot{q}^d(t), u^d(t)\big)\,\mathrm{d}t \\ &\approx hC\big(q^d(t_{k+1/2}), \dot{q}^d(t_{k+1/2}), u^d(t_{k+1/2})\big) \end{aligned}$$

leading to a nonlinear finite dimensional constrained optimization problem..

Discrete Mechanics and Optimal Control (DMOC)

min $J_C^d =$	$\sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, u_k, u_{k+1})$	cost function
s.t.: $q_0 =$	q_b	initial positions
$0 =$	$D_2 L(q_b, \dot{q}_b) + D_1 L_d(q_0, q_1) + F_0^-$	initial momenta
$k =$	$1 \dots N-1$	
$0 =$	$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) + F_{k-1}^+ + F_k^-$	DEL (collision-free)
$0 \leq$	$r_d(q_k, q_{k+1}, u_k, u_{k+1})$	path constraints
$q_N =$	q_e	terminal positions
$0 =$	$D_2 L_d(q_{N-1}, q_N) + F_{N-1}^+ - D_2 L(q_e, \dot{q}_e)$	terminal momenta

..solvable by numerical routines, e.g. SQP, for local extrema.

Incorporating Collisions

DMOC: $q_b, q_e, p_b, p_e \to u_{be}(t)$
opt2: $q_B, q_i, q_E, p_E \to t_i, p_i$

inner optimization: smooth motion from start/collision to collision/end (DMOC)

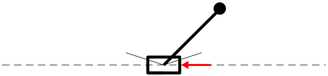
outer optimization: find optimal collision states and times

$$q_B, p_B, t_B \to q_{c1}, p^-_{c1}, t_{c1} \to q_{c2}, p^-_{c2}, t_{c2} \to \dots \to q_{cN}, p^-_{cN}, t_{cI} \to q_E, p_E, t_E$$

input: initial state (q_B, p_B),
terminal state (q_E, p_E) and
switching sequence $(q_{c1}, q_{c2}, \dots, q_{cI})$

output: optimal collision momenta p_{ci} for $i = 1, 2 \dots I$,
optimal collision times t_{ci} for $i = 1, 2 \dots I$ with $t_B < t_{c1} < t_{c2} < \dots < t_E$,
optimal final time t_E,
optimal force $u_i(t)$ for $i = 1, 2 \dots I + 1$ and
trajectory $q_i(t)$ for $i = 1, 2 \dots I + 1$ (verified in simulation)

Pendulum on Cart with Limiters

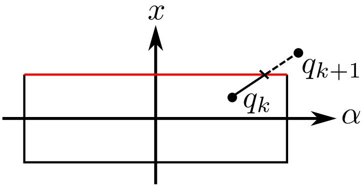


swing-up via one collision

$$
\begin{aligned}
q &= [\alpha, x]^T \\
\mathcal{L} &= \frac{1}{2}\dot{q}^T M \dot{q} - V(q) \\
M &= \begin{bmatrix} m_p l^2 & -m_p l \sin\alpha \\ \text{sym.} & m_p + m_c \end{bmatrix} \\
V &= m_p l g \sin\alpha \\
\delta W^{\mathrm{nc}} &= F \delta x
\end{aligned}
$$

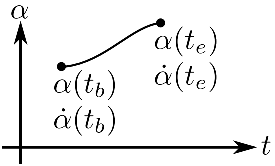
$$
\begin{aligned}
\alpha_{\min} \leq \; & \alpha \leq \alpha_{\max} \\
x_{\min} \leq \; & x \leq x_{\max} \\
F_{\min} \leq \; & F \leq F_{\max}
\end{aligned}
$$

implementation issue #1: find active constraint in simulation



1. detect event by end position outside admissible range,
2. find active constraint from linear interpolation,
3. evaluate collision time step for active constraint.

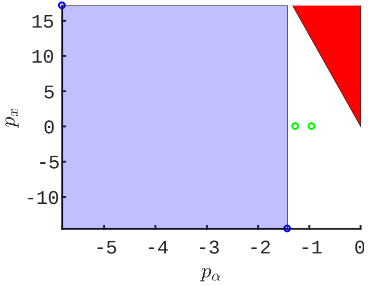
implementation issue #2: find initial guess for DMOC



1. spline approximation $\alpha(t)$ matching I.C. $\alpha(t_b), \dot\alpha(t_b)$ and T.C. $\alpha(t_e), \dot\alpha(t_e)$,
2. determine $x(t)$ from $m_p l \sin\alpha\, \ddot x = J\ddot\alpha + m_p g l \cos\alpha$ with I.C. $x(t_b), \dot x(t_b)$,
3. determine $F(t)$ from $(m_p + m_c)\ddot x - m_p l(\ddot\alpha \sin\alpha + \dot\alpha^2 \cos\alpha) = F(t)$.

This initial guess is in accordance with the equation of motion, but ignores $F_{\min} \leq F \leq F_{\max}$ and $\dot x(t_e)$. Note, $x(t_e)$ does not enter the optimization.

implementation issue #3: exclude unfeasible collisions

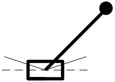


Estimate feasible range of collision states (time, momenta) from

1. velocity condition $\dot{\alpha} < 0$ for collision with $\alpha_{\min}$ and so on,
2. full force motion from initial state to collision and from collision to final state,
3. free motion from initial state to collision and from collision to final state.

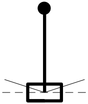
Results

optimal trajectory in the sense of $$J_C = \int_{t_B}^{t_E} \frac{1}{2} F(t)^2 \, \mathrm{d}t$$

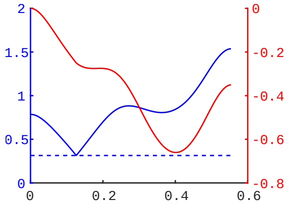


$\alpha = \frac{1}{4}\pi, \mathbf{p} = \mathbf{0}$ $\rightarrow$

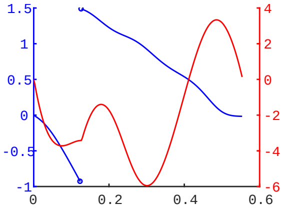
collision ($\alpha = \alpha_{\min} = \frac{1}{10}\pi$) $\rightarrow$



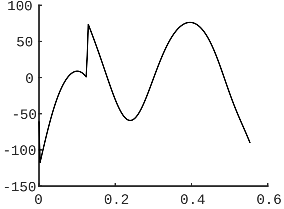
$\alpha = \frac{1}{2}\pi, \mathbf{p} = \mathbf{0}$



positions $\alpha(t)$, $x(t)$



momenta $p_\alpha(t)$, $p_x(t)$



cart force $F(t)$

Current Limitations

- Simulation can not handle yet
  - states of permanent contact with boundary, e.g. resting at limiter.
  - double hits, i.e. two or more collisions per time step or collision with two limiters at a time.
- Optimization of control can hardly handle yet
  - controllability loss for $\alpha = k\pi$, thus restricting limiters to $0 < \alpha_{\min}$ and $\alpha_{\max} < \pi$.
  - full search space, because infeasible points of the outer optimization are excluded by an exceedingly conservative estimation.

Summary

- Variational formulation enables construction of Variational Integrators (VI) and Direct Optimal Control (DMOC) of smooth segments.
- Separations into smooth segments leads to computationally expensive two-layer optimization.
- Works for nonlinear systems with few degrees of freedom and few collisions.
- Finding initial guesses and limits of feasibility is as difficult as optimization itself.

Outlook

- more elegant and efficient formulation of the optimization problem (one-layer)
- equality constraints during collisions (additional holonomic constraints)
- frictional and inelastic collisions
- (global optimization, find optimal switching sequence)

Parameter Values of Example

m_p	$1\,\mathsf{kg}$
m_c	$1\,\mathsf{kg}$
l	$0.5\,\mathsf{m}$
g	$10\,\mathsf{ms}^{-2}$
$\alpha_{\min}$	$\frac{1}{10}\pi$
$\alpha_{\max}$	$\frac{9}{10}\pi$
$x_{\min}$	$-1\,\mathsf{m}$
$x_{\max}$	$1\,\mathsf{m}$
$u_{\min}$	$-10(m_p+m_c)g = -200\,\mathsf{N}$
$u_{\max}$	$10(m_p+m_c)g = +200\,\mathsf{N}$

Parameter Values of Simulation

h	$2e-3$ s
TOL	$1e-14$