Mental Mathematics

calculating without instruments

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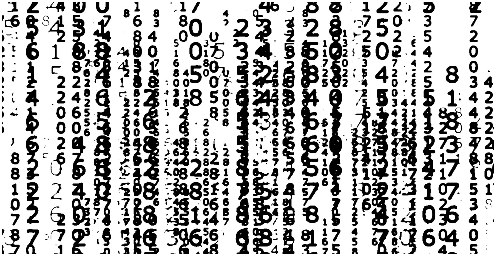
Chemnitz Linux User Group

8th June 2018



Either you are a natural born lightning calculator (Leibniz, Euler) or you known many fast algorithms to choose problem-specifically the best one, instead of having only few general algorithms.

- this talk is mainly limited to numbers with two digits
- derivations of the formulas is simple and left as exercises for the interested audience

Notation:

double bar divides two digits	$a_3 a_2 \| \| a_1 a_0 = a_3 a_2 a_1 a_0,$
single bar divides one digit	$a_1 \| a_0 = a_1 a_0.$

Assumptions:

- most people are able to store a maximum of 7 digits at a time;
- it is hard to keep intermediate results in mind, while performing further calculations;
- it is easier to operate with small numbers $(1, 2, 3)$ than with greater $(7, 8, 9)$.

Requirements for algorithms:

- complex operations are split into simpler ones;
- order of operations such that as few as possible intermediate results need to be stored;
- an approximative solution is earlier found and may serve as start for further iterations.

Outline

Addition & Subtraction

the most basic

right to left

$$
\begin{array}{cccc}
 & 5 & 3 & 8 \\
+ & 3 & 2 & 7 \\
= & 8 & 6 & 5
\end{array}
$$

left to right

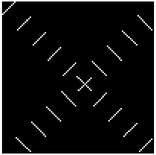
$$538 + 327 = 838 + 27 = 858 + 7 = 865$$

Multiplication

two equal digits, the other two complete to 10

$$
\begin{array}{ccc}
ca_0 \cdot cb_0 & \text{with} & a_0 + b_0 = 10 \\
 & \textbf{or} & \\
a_1 c \cdot b_1 c & \text{with} & a_1 + b_1 = 10
\end{array}
$$

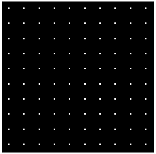
$$
\begin{aligned}
a_1 a_0 \cdot b_1 b_0 &= a_1 \cdot b_1 + c || a_0 \cdot b_0 \\
74 \cdot 34 &= 21 + 4 || 16 = 25\,16 \\
47 \cdot 43 &= 16 + 4 || 21 = 20\,21
\end{aligned}
$$



Multiplication

last digits 5

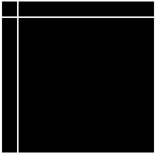
$$
\begin{aligned}
a_1 5 \cdot b_1 5 &= a_1 \cdot b_1 + \frac{a_1 + b_1}{2} \Big|\Big| \begin{cases} 25 & \text{if } a_1 + b_1 = 2k \\ 75 & \text{else} \end{cases} \\
55 \cdot 35 &= 15 + 4.0||25 = 19\,25 \\
65 \cdot 35 &= 18 + 4.5||75 = 22\,75
\end{aligned}
$$



Multiplication

multiplication by 11

$$\begin{aligned} a_1 a_0 \cdot 11 &= a_1 | a_1 + a_0 | a_0 \\ \\ 32 \cdot 11 &= 3|5|2 = 352 \\ 87 \cdot 11 &= 8|15|7 = 957 \end{aligned}$$

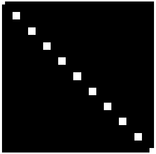


Multiplication

numbers close by

$$(z+a)\cdot(z+b) \quad = \quad z\cdot(z+a+b)+a\cdot b$$

$$107\cdot 108 \quad = \quad 100\cdot 115+56=11556$$



special case middle number

$$(z+a)\cdot(z-a) \quad = \quad z^2-a^2$$

$$61\cdot 59 \quad = \quad 3600-1=3599$$

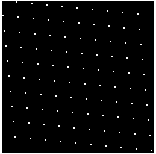
Multiplication

cross multiplication

$$
\begin{aligned}
a_1 a_0 \cdot b_1 b_0 &= a_1 b_1 \cdot 100 + (a_1 b_0 + a_0 b_1) \cdot 10 + a_0 \cdot b_0 \\
23 \cdot 37 &= 6 \cdot 100 + (14 + 9) \cdot 10 + 21 = 851
\end{aligned}
$$

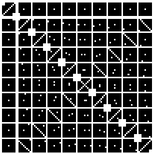
special case *mirror numbers*

$$
\begin{aligned}
ab \cdot ba &= a \cdot b \cdot 100 + (a^2 + b^2) \cdot 10 + a \cdot b \\
57 \cdot 75 &= 35 \cdot 100 + (25 + 49) \cdot 10 + 35 = 4275
\end{aligned}
$$



Multiplication

extensions



So far 25% covered
further coverage by different factorizations, e.g.

$$208 \cdot 53 \quad = \quad 104 \cdot 106 = 100 \cdot 110 + 24 = 11024 \qquad \text{(close by)},$$

friendly intermediate products

$$\begin{aligned} 21 \cdot 13 \quad &= \quad 3 \cdot 7 \cdot 13 = 3 \cdot 91 = 273 \qquad \text{(small factor last)}, \\ 89 \cdot 72 \quad &= \quad 89 \cdot 9 \cdot 8 = 801 \cdot 8 = 6408 \qquad \text{(zero in between)} \end{aligned}$$

or simple extensions

$$98 \cdot 9 = (100 - 2) \cdot 9 = 900 - 18 = 882.$$

Squaring

last digit 5

$$
\begin{aligned}
(a5)^2 &= a \cdot (a+1)||25 \\
45^2 &= 4 \cdot 5||25 = 2025
\end{aligned}
$$

Squaring

last digit 1 or 9

$$
\begin{aligned}
(a0 \pm 1)^2 &= 10 \cdot a^2 \pm 2 \cdot a|1 \\
\\
41^2 &= 160 + 8|1 = 1681 \\
39^2 &= 160 - 8|1 = 1521
\end{aligned}
$$

Squaring

numbers near 50

$$
\begin{aligned}
(50+a)^2 &= 25+a||a^2 \\[1em]
53^2 &= 25+3||09 = 2809 \\
46^2 &= 25-4||16 = 2116
\end{aligned}
$$

Squaring

close by

$$(z+a)^2 = z \cdot (z + 2 \cdot a) + a^2$$

$$74^2 = 70 \cdot 78 + 16 = 5476$$

Squaring

extensions

simple cases $a \leq 11$

$$\begin{aligned} 9^2 &= 81 \\ 90^2 &= 8100 \end{aligned}$$

all together 72%

Division

succession

$$\frac{a}{b} = n \cdot b + R$$

$$\begin{aligned}
\frac{214}{7} &= \mathbf{30} \cdot 7 + 4 \\
\frac{4}{7} &= \mathbf{0.5} \cdot 7 + 0.5 \qquad \frac{40}{7}\,\frac{1}{10} \\
\frac{0.5}{7} &= \mathbf{0.07} \cdot 7 + 0.01 \qquad \frac{50}{7}\,\frac{1}{100}
\end{aligned}$$

$$\ldots$$

$$\frac{214}{7} = \mathbf{30.57}\ldots \qquad (30.571428571)$$

Alternatively, learn basic fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\ldots$ by heart.

Squareroot

average with nearest integer squareroot

$$\sqrt{a} = \frac{1}{2}n + \frac{1}{2}\frac{a}{n}$$

with n^2 being the nearest square number to a, or a guess of it.

$$\begin{aligned}
\sqrt{17} &= \frac{4+4.25}{2} = 4.125 \qquad (4.123105626) \\
\sqrt{30} &= \frac{5+6}{2} = 5.5 \qquad \text{another iteration} \\
&= \frac{5.5+5.45}{2} = 5.475 \qquad (5.477225575)
\end{aligned}$$

Logarithm

minimize required precomputed values by rearrangement

use approximation $\ln(1+x) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$

$$\begin{aligned}
\log_{10} N &= \log_{10}\big((p_1 \cdot p_2 \cdot \dots)(1+x)\big) \\
&= \log_{10} e \cdot \ln\big((p_1 \cdot p_2 \cdot \dots)(1+x)\big) \\
&\approx \log_{10} e \cdot (\ln p_1 + \ln p_2 + \dots + x - \frac{1}{2}x^2 + \dots)
\end{aligned}$$

$$\begin{aligned}
\log_{10} 1211 &= \log_{10}\big(1200 \cdot (1 + 11/1200)\big) \\
&= \log_{10}\big(100 \cdot 12 \cdot (1 + 11/1200)\big) \\
&= 2 + \log_{10}\big(12 \cdot (1 + 11/1200)\big) \\
&= 2 + \log_{10}\big(2^2 \cdot 3 \cdot (1 + 11/1200)\big) \\
&= 2 + \log_{10} e \cdot \ln\big(2^2 \cdot 3 \cdot (1 + 11/1200)\big) \\
&\approx 2 + \log_{10} e \cdot (2\ln 2 + \ln 3 + 11/1200) \\
&\approx 2 + \frac{1}{2.3} \cdot (2 \cdot 0.7 + 1.1 + 0.01) \\
&\approx 3.1 \qquad\qquad (3.083144143)
\end{aligned}$$

Interest Rate

doubling time of investments

Years for doubling at an annual interest rate of $P\% < 25\%$

$$y = 70/P \qquad (y = \log_{1+p} 2) \qquad \text{with} \qquad (p = P/100)$$

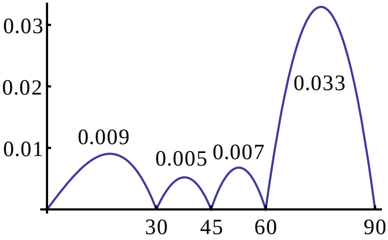
$$P = 5\% \qquad \rightsquigarrow \qquad y = \frac{70}{5} = 14 \qquad (14.2066990)$$

Sine & Co. I

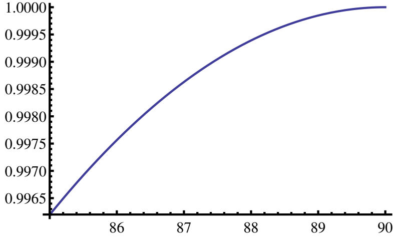
linear interpolation and visual correction

linear interpolation

x	0°	30°	45°	60°	90°
$\sin(x)$	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{2}}{2} \approx \frac{707}{1000}$	$\frac{\sqrt{3}}{2} \approx \frac{866}{1000}$	$\frac{\sqrt{4}}{2} = 1$



sine minus linear interpolation



sine close to maximum

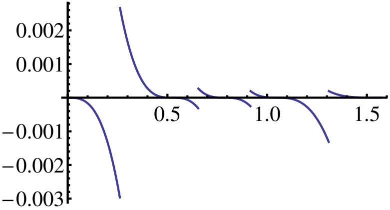
$$\sin(57^\circ) \quad = \quad 0.834 + 0.004 = 0.838 \qquad (0.838671)$$

Sine & Co. II

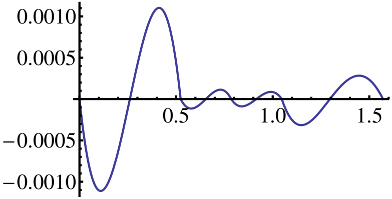
alternatives

Taylor expansion around $x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$

$$\sin(x + \Delta x) \approx \sin x + \Delta x \cos x - \frac{(\Delta x)^2}{2} \sin x$$



taylor expansion error



polynomial fit error

piecewise polynomial fit reduces error further ($\varepsilon \approx 10^{-3}$) with same computational effort but more precomputed values (coefficients)

$$\sin(x + \Delta x) \approx a_2(\Delta x)^2 + a_1 \Delta x + \sin x$$

Verification

casting out 9s for addition and multiplication (division)

$$
\begin{aligned}
\operatorname{mod}(a,9) &\quad + \operatorname{mod}(b,9) &= \operatorname{mod}(a+b,9)\\
\operatorname{mod}(a,9) &\quad \cdot \operatorname{mod}(b,9) &= \operatorname{mod}(a\cdot b,9)
\end{aligned}
$$

$$
\operatorname{mod}(a_n a_{n-1} a_{n-2} \ldots a_2 a_1 a_0, 9) = \sum_{i=0}^{n} a_i \qquad \text{sum of digits}
$$

$$
\begin{aligned}
21 + 31 &\equiv 52 &\mod 9\\
(2+1) + (3+1) &\equiv (5+2) &\mod 9\\
7 &\equiv 7 &\mod 9
\end{aligned}
$$

$$
\begin{aligned}
21 \cdot 31 &\equiv 651 &\mod 9\\
(2+1) \cdot (3+1) &\equiv 6+5+1 &\mod 9\\
12 &\equiv 12 &\mod 9\\
3 &\equiv 3 &\mod 9
\end{aligned}
$$

Verification

casting out 11s, 99s or 101s

$$
\begin{aligned}
21 \cdot 31 &\equiv 651 && \bmod 11\\
(-2+1)\cdot(-3+1) &\equiv 6-5+1 && \bmod 11\\
2 &\equiv 2 && \bmod 11
\end{aligned}
$$

$$
\begin{aligned}
229 \cdot 721 &\equiv 165109 && \bmod 99\\
(02+29)\cdot(07+21) &\equiv 16+51+09 && \bmod 99\\
868 &\equiv 76 && \bmod 99\\
08+68 &\equiv 76 && \bmod 99
\end{aligned}
$$

$$
\begin{aligned}
229 \cdot 721 &\equiv 165109 && \bmod 101\\
(-02+29)\cdot(-07+21) &\equiv 16-51+09 && \bmod 101\\
378 &\equiv -26 && \bmod 101\\
(-03+78) &\equiv 75 && \bmod 101
\end{aligned}
$$

Guesstimation

if moderately accurate solutions are sufficient

$$8\,367 + 5\,819 \approx 8\,000 + 6\,000 = 14\,000 \qquad (14\,186)$$

$$1.39 + 0.87 + 2.46 \approx 1.50 + 1.00 + 2.50 = 5.00 \qquad (4.72)$$

$$\frac{5\,000\,000}{365} = \frac{500}{365} 10\,000 \approx 1.4 \cdot 10\,000 \qquad (13\,698.63)$$

$$73 \cdot 65 \approx \begin{cases} 70 \cdot 68 = 4760 \\ 78 \cdot 60 = 4680 \end{cases} \qquad (4745)$$

The fun starts here

- Summary
  - mental math - elementary operations [1]
    In order to extend the small 1x1 to the big 1x1, you need to learn special cases, how to recognize and rearrange them quickly.
- Outlook
  - memory training [2]
  - mental math - advanced calculus [3, 4]

shell script

```
#!␣/bin/bash

now=$(date)
exercises=$(~/projects/krt/build/krt␣F)
thunderbird␣-compose
"from=sender@xyz.com,
␣to=receiver@xyz.com,
␣subject='$now',body='$exercises'"
```

References

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