Dynamics of a rotor partially filled with a viscous incompressible Fluid

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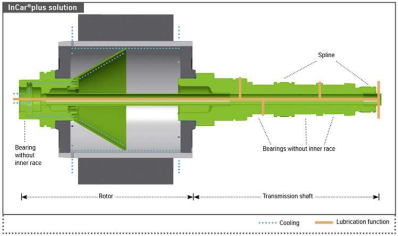
GAMM 2016 Section S05 – Nonlinear Oscillations

Motivation

[Ibrahim 2005], [Gasch 2012]

Rotors filled with liquids are of technical relevance for

- centrifuges,
- liquid-filled projectiles,
- drives and turbines with inner cooling.



E-drive with active cooling inside

https://incarplus.thyssenkrupp.com

Scope

rotor partially filled with a liquid

goal: minimal model for simulation and control

Literature

Warning! This problem and its results (in some variations) are not new!

Kollmann 1962: *Experimentelle und theoretische...*

Moiseev, Rumnyantsev 1968: *Dynamic Stability of Bodies...*

Hendricks, Morton 1979: *Stability of a Rotor partially...*

Ibrahim 2005: *Liquid Slooshing Dynamics*

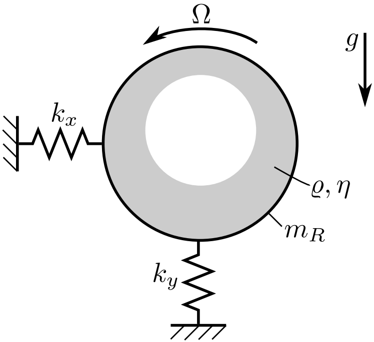
Brommundt, Ostermeyer 1986: *Stabilität eines fliegend...*

Keisenberg, Ostermeyer 2015: *Synchronization effects...*

...

However, the methodolgy developed here should be new (in some way) and be extendable to more complex models.

Model

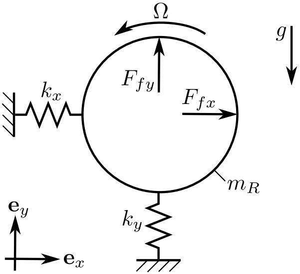


planar model of

- rigid circular rotor, elastically mounted;
- partially filled with an incompressible, viscous liquid;
- rotating with prescribed angular velocity $\Omega(t) > \sqrt{\frac{g}{R_a}}$;
- and subjected to gravitational field.

Rotor equations

rotor model

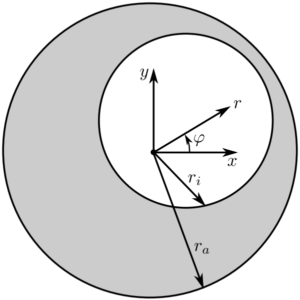


$$0 = m_R \ddot{u}_x + k_x u_x - l \int_0^{2\pi} p(r_a, \varphi) \cos\varphi \, r_a \mathrm{d}\varphi$$

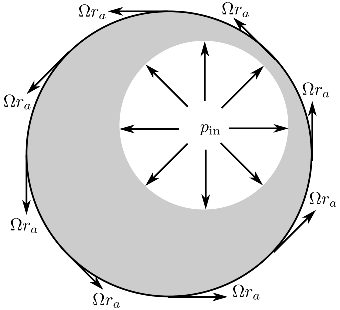
$$0 = m_R \ddot{u}_y + k_y u_y - l \int_0^{2\pi} p(r_a, \varphi) \sin\varphi \, r_a \mathrm{d}\varphi$$

Fluid equations I

coordinate system



boundary conditions



Navier Stokes equation for incompressible fluids

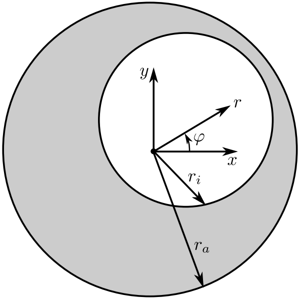
$$0 = \varrho\left(\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}\right) - \varrho\mathbf{b} + \nabla p - \eta\Delta\mathbf{v}$$

B.C.: $v_r(t, r_a, \varphi) = 0, \quad v_\varphi(t, r_a, \varphi) = \Omega r_a, \quad p(t, r_i, \varphi) = p_{\rm in}$

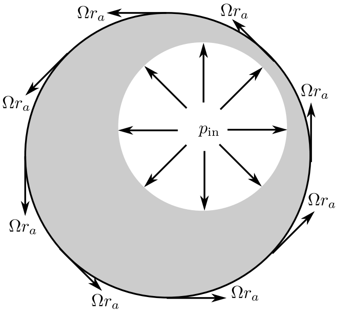
I.C.: $v_r(0, r, \varphi) = v_{r0}(r, \varphi), \quad v_\varphi(0, r, \varphi) = v_{\varphi 0}$

Fluid equations II

coordinate system



boundary conditions



Evolution of the free boundary

$$\frac{\partial r_i(t,\varphi)}{\partial t} + \mathbf{v} \cdot \nabla r_i = v_r(t, r_i, \varphi)$$

I.C.: $r_i(0,\varphi) = r_{i0}(\varphi)$

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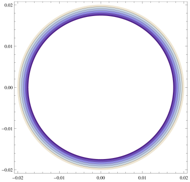
Overview

	unknows	**equations**
rotor	$u_x(t), \dot{u}_x(t), u_y(t), \dot{u}_y(t)$	4 ode (state space)
fluid	$v_r(t, r, \varphi), v_\varphi(t, r, \varphi)$	2 pde (N.S.eq. r, φ)
fluid	$r_i(t, \varphi)$	1 pde (F.B.eq.)
	7	**7**

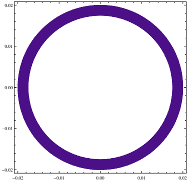
$$\textbf{fluid} \underset{\text{no-slip condition}}{\overset{\text{pressure}}{\rightleftarrows}} \textbf{rotor}$$

Approximation of the fluid velocity field

- Firstly, circumferential component $v_\varphi \approx \tilde{v}_\varphi$
- $\tilde{v}_r$ is solution of ode $\frac{\partial(r\tilde{v}_r)}{\partial r} = -\frac{\partial \tilde{v}_\varphi}{\partial \varphi}$ (incompressibility)



$\tilde{v}_\varphi$



$\tilde{v}_r$

$$\tilde{v}_{\varphi R}(t,r,\varphi) = \Omega r$$
$$\tilde{v}_{\varphi 0}(t,r,\varphi) = (r_a - r)\, V_0(t)$$
$$\tilde{v}_{\varphi 1}(t,r,\varphi) = (r_a - r)\big(V_{1s}(t)\sin\varphi + V_{1c}(t)\cos\varphi\big)$$

$$\tilde{v}_{rR}(t,r,\varphi) = 0$$
$$\tilde{v}_{r0}(t,r,\varphi) = 0$$
$$\tilde{v}_{r1}(t,r,\varphi) = \frac{1}{2r}(r - r_a)^2\big(V_{1s}(t)\cos\varphi - V_{1c}(t)\sin\varphi\big)$$

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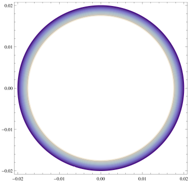
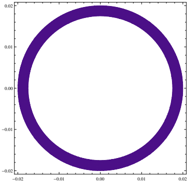


$\tilde{v}_\varphi$



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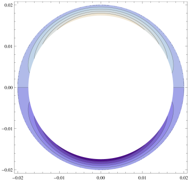
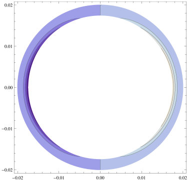


$\tilde{v}_\varphi$



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Approximation of the free boundary

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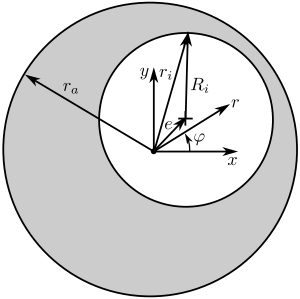
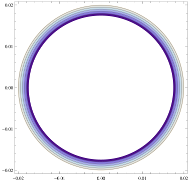


free boundary is approximated by an excentric circle $\left(e(t) = \sqrt{e_x(t)^2 + e_y(t)^2} < R_a - R_i\right)$ of radius R_i

$$\begin{aligned} r_i(t,\varphi) &= e_x(t)\cos\varphi + e_y(t)\sin\varphi \\ &\quad + \sqrt{R_i^2 - \left(e_x(t)\sin\varphi - e_y(t)\cos\varphi\right)^2} \end{aligned}$$

Approximation of the fluid pressure field



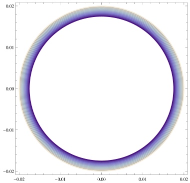
$$\tilde{p}_R(t,r,\varphi) = p_{in} - \frac{1}{2}\varrho\Omega^2\left(r^2 - R_i^2 + e(t)^2\right) + \varrho\Omega^2 r\left(e_x(t)\cos\varphi + e_y(t)\sin\varphi\right)$$

with $e_x = -\frac{\ddot{u}_x}{\Omega^2}$ $\quad e_y = -\frac{\ddot{u}_y + g}{\Omega^2}$

$$\tilde{p}_0(t,r,\varphi) = (r - r_i)P_0(t)$$

$$\tilde{p}_1(t,r,\varphi) = (r - r_i)\left(P_{1s}(t)\sin\varphi + P_{1c}(t)\cos\varphi\right)$$

Approximation of the fluid pressure field



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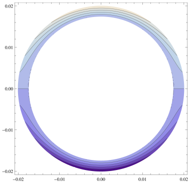

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Approximation of the PDE solution

Eight unknowns for the fluid field
$V_0(t), V_{1s}(t), V_{1c}(t), P_0(t), P_{1s}(t), P_{1c}(t), e_x(t), e_y(t).$

Eight ODEs by method of weighted residuals

$$0 = \int_0^{2\pi}\int_{r_i}^{r_a} \mathsf{NSE}_r \left\{ \begin{array}{l} \cdot 1 \\ \cdot \sin\varphi \\ \cdot \cos\varphi \end{array} \right. r\mathrm{d}\varphi\mathrm{d}r$$

$$0 = \int_0^{2\pi}\int_{r_i}^{r_a} \mathsf{NSE}_\varphi \left\{ \begin{array}{l} \cdot 1 \\ \cdot \sin\varphi \\ \cdot \cos\varphi \end{array} \right. r\mathrm{d}\varphi\mathrm{d}r$$

$$0 = \int_0^{2\pi} \mathsf{FBE} \left\{ \begin{array}{l} \cdot \sin\varphi \\ \cdot \cos\varphi \end{array} \right. r\mathrm{d}\varphi\mathrm{d}r$$

Simplifications

Assuming $e/R_i \ll 1$

$$\int_0^{2\pi}\int_{r_i}^{r_a} \mathsf{NSE}\; r\mathrm{d}\varphi\mathrm{d}r \approx \int_0^{2\pi}\int_{R_i}^{r_a} \mathsf{NSE}\; r\mathrm{d}\varphi\mathrm{d}r$$

$$(r-r_i)P_0(t) \approx (r-R_i)P_0(t)$$

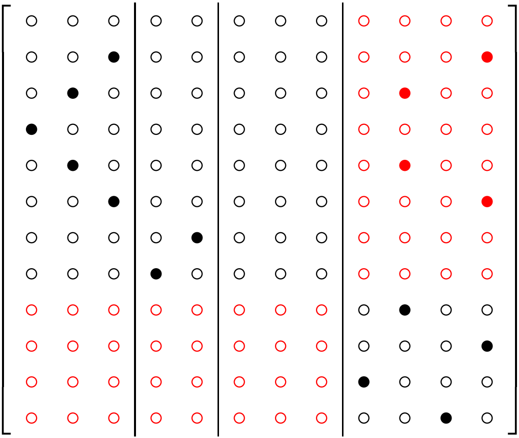
$$(r-r_i)(P_{1s}(t)\dots) \approx (r-R_i)(P_{1s}(t)\dots)$$

$$r_i(t,\varphi) \approx R_i + e_x(t)\cos\varphi + e_y(t)\sin\varphi$$

Total system

DAE $\quad \mathbf{M}\dot{\mathbf{z}} = \mathbf{L}\mathbf{z} + \mathbf{n} + \mathbf{c}$

$\mathbf{M}\dot{\mathbf{z}} =$



$$\begin{bmatrix} \dot{V}_0 \\ \dot{V}_{1s} \\ \dot{V}_{1c} \\ \hline \dot{e}_x \\ \dot{e}_y \\ \hline \dot{P}_0 \\ \dot{P}_{1s} \\ \dot{P}_{1c} \\ \hline \dot{u}_x \\ \ddot{u}_x \\ \dot{u}_y \\ \ddot{u}_y \end{bmatrix}$$

Total system

$$\text{DAE} \qquad \mathbf{M}\dot{\mathbf{z}} = \mathbf{L}\mathbf{z} + \mathbf{n} + \mathbf{c}$$

$$\mathbf{L}\mathbf{z} =$$

$$\left[\begin{array}{ccc|cc|ccc|cccc}
\bullet & \circ & \circ & \circ & \circ & \bullet & \circ & \circ & {\color{red}\circ} & {\color{red}\circ} & {\color{red}\circ} & {\color{red}\circ} \\
\circ & \bullet & \bullet & \circ & \bullet & \circ & \bullet & \circ & {\color{red}\circ} & {\color{red}\circ} & {\color{red}\circ} & {\color{red}\circ} \\
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{\color{red}\circ} & {\color{red}\circ} & {\color{red}\circ} & {\color{red}\circ} & {\color{red}\circ} & {\color{red}\circ} & {\color{red}\circ} & {\color{red}\circ} & \circ & \circ & \circ & \bullet
\end{array}\right]
\left[\begin{array}{c}
V_0 \\ V_{1s} \\ V_{1c} \\ \hline e_x \\ e_y \\ \hline P_0 \\ P_{1s} \\ P_{1c} \\ \hline u_x \\ \dot{u}_x \\ u_y \\ \dot{u}_y
\end{array}\right]$$

Total system

DAE $\quad \mathbf{M}\dot{\mathbf{z}} = \mathbf{L}\mathbf{z} + \mathbf{n} + \mathbf{c}$

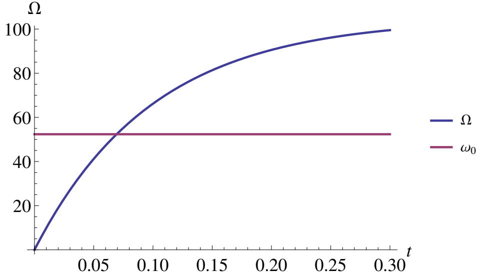
$$\mathbf{n} + \mathbf{c} = \begin{bmatrix} \bullet V_0^2 + \bullet V_{1s}^2 + \bullet V_{1c}^2 \\ \bullet V_0 V_{1s} \\ \bullet V_0 V_{1c} \\ \circ \\ \bullet V_0 V_{1c} \\ \bullet V_0 V_{1s} \\ \bullet V_0 e_x \\ \bullet V_0 e_y \\ \circ \\ \circ \\ \circ \\ \circ \end{bmatrix} + \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \circ \\ \bullet \\ \bullet \\ \circ \\ \circ \\ \bullet \\ \bullet \\ \circ \\ \circ \end{bmatrix}$$

Example

[Hindmarsh et al. 2005]

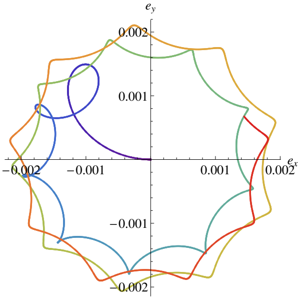
DAE is solved numerically with the *IDA algorithm* from *SUNDIALS*. Here for a run-up prescribed by the angular velocity

$$\Omega(t) = 2\omega_0(1 - e^{-t/\tau}).$$

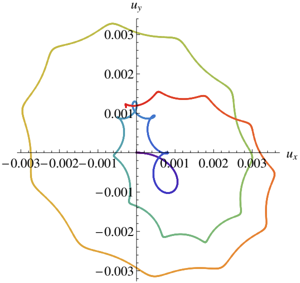


angular velocity during run-up ($\tau = 0.1$s)

Fluid and rotor motion



Excentricity



Rotor displacement ($t = 0 \ldots 0.3$s)

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Velocity field

$v_\varphi(t = 0 \ldots 0.03\mathsf{s}, r, \varphi)$

$v_r(t = 0 \ldots 0.03\mathsf{s}, r, \varphi)$

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Pressure field

$$p(t = 0 \ldots 0.03\mathsf{s}, r, \varphi)$$

Summary and Outlook

The **minimum model** for a fluid filled and elastically mounted rotor, obtained in this contribution, is a

nonlinear DAE of dimension 12,

which is small enough for inclusion into simulations of composed systems and controller design.

Future work is related with

- verification (limit cases, literature results),
- further analysis types (modal, stability, bifurcations),
- algorithmic speed-up, enable large slooshing,
- 3D model (rotor as beam, bearing models).

Input data

model parameters			initial conditions		
R_a	0.02	m	$V_0(0)$	0	(m/s)/m
R_i	0.0175	m	$V_{1s}(0)$	0	(m/s)/m
p_{in}	10^5	Pa	$V_{1c}(0)$	0	(m/s)/m
ϱ	1000	kg/m^3	$e_x(0)$	0	m
η	$550 \cdot 10^{-6}$	Pa s	$e_y(0)$	0	m
g	9.81	m/s	$u_x(0)$	0	m
l	0.5	m	$\dot{u}_x(0)$	0	m/s
k_x	11192.9	N/m	$u_y(0)$	0	m
k_y	11192.9	N/m	$\dot{u}_y(0)$	0	m/s
m_R	3.93	kg			

Appendix

Navier Stokes equation in polar coordinates for an incompressible fluid

$$
\begin{aligned}
0 &= \varrho\left(\frac{\partial v_r}{\partial t} + \frac{\partial v_r}{\partial r} v_r + \frac{\partial v_r}{\partial \varphi}\frac{v_\varphi}{r} - \frac{v_\varphi^2}{r}\right) - \varrho b_r + \frac{\partial p}{\partial r} \\
&\quad -\eta\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \varphi^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi}\right) \\
0 &= \varrho\left(\frac{\partial v_\varphi}{\partial t} + \frac{\partial v_\varphi}{\partial r} v_r + \frac{\partial v_\varphi}{\partial \varphi}\frac{v_\varphi}{r} + \frac{v_r v_\varphi}{r}\right) - \varrho b_\varphi + \frac{1}{r}\frac{\partial p}{\partial \varphi} \\
&\quad -\eta\left(\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial v_\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial \varphi^2} - \frac{v_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi}\right)
\end{aligned}
$$