A simple time-stepping scheme for the bouncing ball example

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There are two basic categories for numerical integration of nonsmooth dynamical systems: event-driven and time-stepping. Since MATLAB offers a nice tutorial for an event-driven simulation of a bouncing ball, we deliver the missing part, a time-stepping scheme for the same model in accordance with the K.I.S.S. principle. This scheme turns out robust and withstands ZENO's paradox.

1 Introduction

Nonsmooth systems are characterized by two correlated features, a nonsmooth evolution with respect to time and a set of nonsmooth laws constraining the state. These systems are frequently used to model contacts in mechanics, i.e. impact and friction. Their numerical simulation is divided into two basic categories: event-driven and time-stepping. The former is more precise and enables higher order methods, but it is limited to few events; whereas the latter is more robust and allows for many events.

Event-driven schemes are simply structured (not claiming their efficient implementation is easy). Events are detected by the root finding of a switching function and then the integration is restarted. MATLAB contains a nice tutorial for the event-driven simulation of a bouncing ball, just type `doc ballode`.

Time-stepping schemes are sort of more *complex*, as they handle the discrete-time events in an integral sense, i.e. the integrator marches through time and does not care, when exactly the events happen. Of course there are many excellent textbooks [1, 2], but to gather the implementation from the sophisticated mathematical notation (differential inclusions, variational inequalities) may still pose problems. So let's get started with the simple example of a ball.

2 Model Description

The ball is subject to a continuous-time force f, which may represent gravity and external forcing and the potential reaction force λ of the ground. It is moving in a one-dimensional domain bounded below by the ground. Mathematically its dynamics are governed by

$$
\begin{aligned}
m\ddot{y}(t) &= f(t) + \lambda(t), && \text{(1a)}\\
y(0) &= y_0 \geq 0, && \text{(1b)}\\
\dot{y}(0^-) &= \dot{y}_0, && \text{(1c)}\\
0 \leq y(t) &\perp \lambda(t) \geq 0, && \text{(1d)}\\
\dot{y}(t^+) &= -\varepsilon \dot{y}(t^-), \qquad \text{if } \; y(t) = 0 \; \text{ and } \; \dot{y}(t^-) < 0. && \text{(1e)}
\end{aligned}
$$

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The equation of motion (1a) is nothing else than NEWTON's law, the initial conditions (1b)-(1c) comply with the unilateral constraint $y \geq 0$. The mysterious symbol $\perp$ in the complementarity condition (1d) states that either the contact is closed $y = 0$ or the contact force is inactive $\lambda = 0$, in addition this line tells us that height and contact force are bound to nonnegative values $y \geq 0$ and $\lambda \geq 0$, respectively. The impact law (1e) characterizes the contact by the coefficient of restitution as perfectly elastic $\varepsilon = 1$, inelastic $\varepsilon = 0$ or some real-world value in between[1].

The system of equations and inequalities (1) allows two contact regimes: the impact of zero-duration (bounce back) and resting on the ground (static equilibrium).

3 Time Discretization

We present a MOREAU time-stepping scheme [4] deprived of tuning parameters. Since the problem is one-dimensional, we do not care about any tangent or normal cones, there is only one direction. Assuming $y_k > 0$, we start with an EULER-forward step for the position

$$y_{k+1} = y_k + h\dot{y}_k, \tag{2}$$

where $h > 0$ denotes the time step. If the ball remains in free flight $y_{k+1} > 0$, then we make an EULER-backward step for the velocity

$$\begin{aligned} m(\dot{y}_{k+1} - \dot{y}_k) &= hf_{k+1}, & \text{(3a)} \\ \lambda_{k+1} &= 0, & \text{(3b)} \end{aligned}$$

while there is no contact. Note that time-stepping schemes deal rather with impulses than with forces, meaning the integral of a force over a time step is relevant and not its time-profile.

If $y_{k+1} \leq 0$ indicates a contact, then we have to solve the linear complementarity system

$$\begin{aligned} m(\dot{y}_{k+1} - \dot{y}_k) &= hf_{k+1} + h\lambda_{k+1}, & \text{(4a)} \\ 0 \leq \dot{y}_{k+1} + \varepsilon\dot{y}_k \quad &\perp \quad h\lambda_{k+1} \geq 0. & \text{(4b)} \end{aligned}$$

There exist powerful LCP-solvers [3] to solve for velocity $\dot{y}_{k+1}$ and contact impulse $h\lambda_{k+1}$. However, this LCP we can resolve by hand. From equation (4a) we find the limit condition for contact $h\lambda_{k+1} = 0$, i.e. opening or closing. If the external force f is strong enough to fulfill a velocity change as the impact law (1e) dictates $-m(1+\varepsilon)\dot{y}_k = hf_{k+1}$, or even more $-m(1+\varepsilon)\dot{y}_k < hf_{k+1}$, then the contact is inactive and the ball moves as in free flight. This is a consequence of the restriction $\lambda \geq 0$, that the ground can not pull. Otherwise the impact law needs support from the contact force $h\lambda_{k+1} = -m(1+\varepsilon)\dot{y}_k - hf_{k+1}$.

Finally, here comes the source code, providing all parameter values, and the corresponding plots.

Figure 1 shows a ball subjected to gravitation, which is dropped from a certain height and comes to rest in finite time via an infinite number of bounces. The integrator passes this accumulation point, close to $t = 6\,\mathrm{s}$, and enters a state of rest, in which the contact force is in equilibrium with the weight.

Figure 2 shows position and contact force of a ball which is subjected to an increasing external force in addition to gravity. The complementary nature of contact gap and force is evident.

[1] We peacefully ignore explosive contact and full penetration.

Listing 1: simulation of a bouncing ball (MATLAB)

```
% Implementation of a time-stepping scheme [Moreau1988]
% for a bouncing ball (1D), ground modeled by unilateral constraint
% y>=0 and coefficient of restitution dydt_plus=-rc*dydt_minus
clear; clc; close all;

N=1000; h=0.01; T=N*h; t=0:h:T; % time discretization
rc=0.9;       % coefficient of restitution
m=1; g=10;        % mass and gravitational acceleration
y=zeros(N+1,1); v=zeros(N+1,1); % position and velocity arrays
R=zeros(N,1); % reaction impulse, value at t=0 depends on past
F=zeros(N,1); % external impulse, value at t=0 does not enter

% i.c. and forcing for bouncing --> rest "contact closing" (fig.1)
q0=0.5; v0=0; % initial conditions
f=@(t) -m*g; % external force (continuous-time)
% a sum of a geometric series gets a physical meaning
zeno_time=sqrt(8*q0/g)/(1-rc)-sqrt(2*q0/g); %assuming v0=0

% i.c. and forcing for resting --> flight "contact opening" (fig.2)
%q0=0.0; v0=0; % initial conditions
%f=@(t) (-1.2*sin(t*2*pi/T)-1)*m*g; % external force (cont.-time)

y(1)=q0; v(1)=v0;
for n=1:N

    % Euler-forward for position
    y(n+1)=y(n)+v(n)*h;

    % Euler-backward for velocity
    F(n)=h*f(t(n+1));% integrated force at t=t(n+1)
    v(n+1)=v(n)+F(n)/m; % free flight, otherwise overwritten by LCP

    % LCP for contact, manually resolved
    if (y(n+1)<=0)     % prevent penetration
        dv=-(1+rc)*v(n); % (potential) velocity jump
        if (m*dv)>F(n) % ground can only push, but not pull
            v(n+1)=v(n)+dv; % impact law
            R(n)=m*dv-F(n);   % reaction force from eq. of motion
        end % else nothing to do
    end % else nothing to do

end

figure;
yyaxis left; plot(t,y, [zeno_time, zeno_time],[-0.1, 0.6],'k--');
xlabel('time [s]'); ylabel('position [m]'); ylim([-0.1 0.6])
yyaxis right; plot(t(2:end), R); % R at t=0 is not determined
ylabel('reaction impulse [Ns]'); ylim([-1 6]);
```

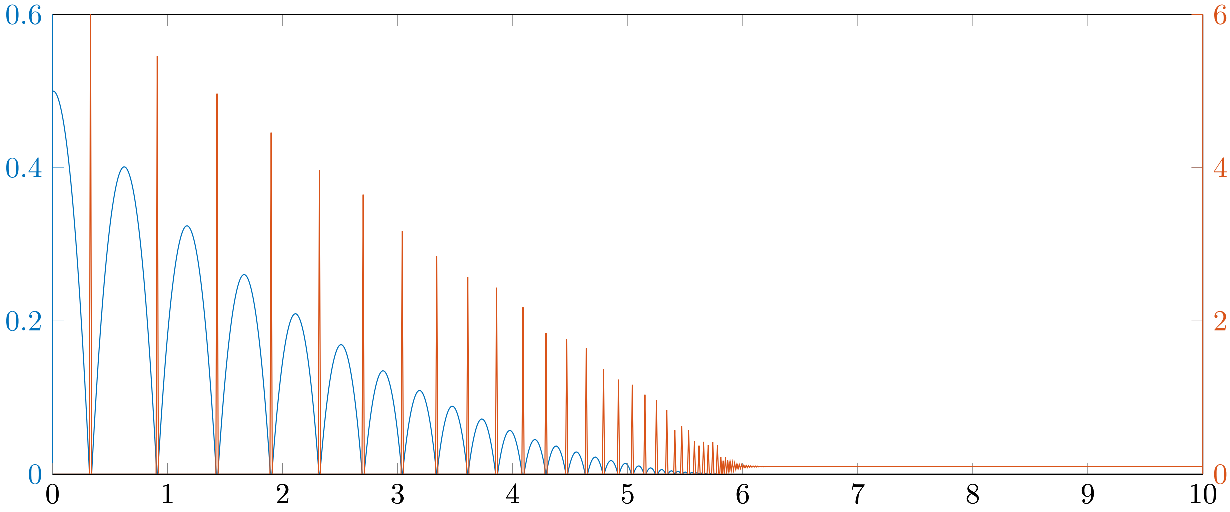

Figure 1: Position y (blue) and contact impulse $h\lambda$ (red) of the *landing*.

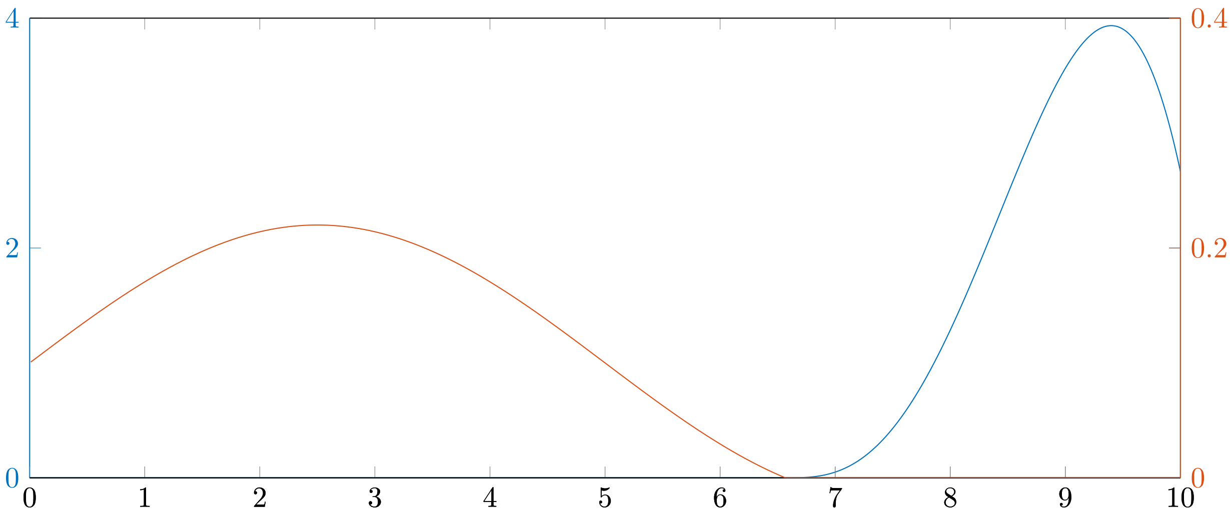

Figure 2: The *take-off* is due to an additional force opposed to gravitation (y, $h\lambda$).

References

[1] Vincent Acary and Bernard Brogliato. *Numerical methods for nonsmooth dynamical systems: applications in mechanics and electronics*. Springer Science & Business Media, 2008. `doi:10.1007/978-3-540-75392-6`.

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[4] Jean J. Moreau. Unilateral contact and dry friction in finite freedom dynamics. In *Nonsmooth mechanics and Applications*, pages 1–82. Springer, 1988. `doi:10.1007/978-3-7091-2624-0_1`.